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ON HAMILTON 'S PRINCIPLE FOR NONHOLONOMIC SYSTEMS

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The conditions under which the three forms of Hamilton 's variational principle were derived for nonholonomic systems by Holder [1], Voronets [2], and Suslov [3] are analyzed in the general case of nonlinear and, also, in particular cases of linear relationships. It is shown that these three forms are equivalent and transformable into one another. Gererally Hamilton 's principle in relation to nonholonomic systems is not the principle of stationary action. although under specific conditions of real motion of such systems it can be found among solutions of Euler 's equations of the Lagrange variational problem. The conditions under which Hamilton's principle applied to related motions of a nonholonomic system has the characteristics of the principle of stationary motion are derived. This is closely related to the question of applicability to nonholonomic systems of the generalized Hamilton — Jacobi method of integrating the equations of motion [4]. The necessary and sufficient conditions of that method applicability to nonholonomic systems have been found to be equivalent to the conditions noted above [5]. It is shown that the method is applicable then and only then when Hamilton 's principle can be treated as the principle of stationary action. Examples are presented.

1. Let us consider a system of material points whose independent Lagrangian coordinates we denote by q_i $(i = 1, \ldots, n)$. Let the system be subjected to forces defined by the force function $U(q_i, t)$, and constrained by ideal nonintegrable relationships of the form

$$f_l(q_i, q_i^{\cdot}, t) = 0 \quad (l = 1, \ldots, r < n) \tag{1.1}$$

which are generally nonlinear with respect to the generalized velocities $q_i^{\cdot} \equiv dq_i / dt$ $(i = 1, \ldots, n)$ where t denotes time. Relationships (1.1) are assumed independent, hence

$$\operatorname{rank} \left\| \frac{\partial f_l}{\partial q_i^{\cdot}} \right\| = r \tag{1.2}$$

Equations (1.1) with conditions (1.2) can be solved for some r dependent velocities and represented, for instance, in the form

$$f_l(q_i, q_i^{\cdot}, t) = q_{k+l}^{\cdot} - \varphi_l(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_k^{\cdot}, t) = 0 \tag{1.3}$$

where the velocities $q_s^{\cdot}$ $(s = 1, \ldots, k = n - r)$ are assumed independent.

We write the general equation of dynamics which defines the d ' Alambert — Lagrange principle as

$$\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}}\right)\delta q_i=0 \tag{1.4}$$

where $L(q, q^{\cdot}, t) = T + U$ is the Lagrange function, $T(q, q^{\cdot}, t)$ is the kinetic energy, and δq_i are possible (virtual) displacements that satisfy Chetaev's conditions

$$\sum_{i=1}^{n}\frac{\partial f_l}{\partial q_i^{\cdot}}\delta q_i=0 \quad (l=1,\ldots,r) \tag{1.5}$$

For relationships of the form (1.3) these conditions are of the form

$$\delta q_{k+l}=\sum_{s=1}^{k}\frac{\partial \varphi_l}{\partial q_s^{\cdot}}\delta q_s \quad (l=1,\ldots,r) \tag{1.6}$$

Hamilton's principle can be obtained by integrating Eq. (1.4) with respect to t within some constant limits t_0 and t_1

$$\int_{t_0}^{t_1}\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}}\right)\delta q_i\,dt=0$$

on the assumption that the belonging to class C^2 functions δq_i satisfy conditions

$$\delta q_i=0 \quad \text{for} \quad t=t_0,\ t_1 \quad (i=1,\ldots,n) \tag{1.7}$$

Integrating in the preceding equality by parts the terms of the form

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}}\delta q_i$$

with allowance for conditions (1.7) we reduce that equality to

$$\int_{t_0}^{t_1}\sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial q_i^{\cdot}}\frac{d}{dt}\delta q_i\right)dt=0 \tag{1.8}$$

in which, unlike in (1.4), we have the time derivatives of coordinate variations constrained by conditions (1.5). Since the latter do not uniquely define δq_i, there exists evidently some arbitrariness in the determination of derivatives of δq_i. Two equivalent points of view exist in analytic mechanics on the relation of these derivatives with variations δq_i of generalized velocities [6].

According to the first which belongs to Hölder [1] and is based on rules of the calculus of variation the following commutation relationships

$$\frac{d}{dt}\delta q_i=\delta q_i^{\cdot} \quad (i=1,\ldots,n) \tag{1.9}$$

are valid for all coordinates. With this definition of $\delta q_i^{\cdot}$ the variation of functions (1.1) over possible permutations, with (1.5) taken into account, are of the form

$$\delta f_l = \sum_{i=1}^{n} \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt} \frac{\partial f_l}{\partial q_i^{\cdot}} \right) \delta q_i \quad (l = 1, \ldots, r) \tag{1.10}$$

If formulas (1.1) are integrable, expressions (1.10) are identically zero, and if they are not integrable then, although not identically zero, they may, if they are nonlinear, become zero on the strength of the system equations of motion [7].

Note that the identities

$$\delta f_l = 0 \quad (l = 1, \ldots, r) \tag{1.11}$$

and conditions(1.9) are compatible when Eqs. (1.1) are integrable, i.e. in the case of holonomic systems.

For relationships of the form (1.3) formulas (1.10) assume the form

$$\delta f_l = \delta q_{k+l}^{\cdot} - \delta \varphi_l = \sum_{s=1}^{k} A_s^{k+l} \delta q_s \quad (l = 1, \ldots, r) \tag{1.12}$$

where allowance is made for equalitites (1.6) and the following notation is used:

$$A_s^{k+l} = \frac{d}{dt} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} - \frac{\partial \varphi_l}{\partial q_s} - \sum_{\nu=1}^{r} \frac{\partial \varphi_l}{\partial q_{k+\nu}} \frac{\partial \varphi_\nu}{\partial q_s^{\cdot}} \tag{1.13}$$

When by virtue of the equations of motion of a nonholonomic system $\delta f_l = 0$, then, as implied by (1.12), all $A_s^{k+l} = 0$ $(s = 1, \ldots, k)$, and vice versa.

According to the second point of view propounded by Appel and Suslov [3] the identities (1.11) are valid, and this implies that formulas (1.9) are only valid for the independent velocities

$$\frac{d}{dt} \delta q_s = \delta q_s^{\cdot} \quad (s = 1, \ldots, k) \tag{1.14}$$

Expressions for the variation of dependent velocities $q_{k+l}^{\cdot}$ $(l = 1, \ldots, r)$, defined by Eqs. (1.3) are obtained from conditions (1.11) in the form of equalities

$$\bar{\delta} q_{k+l}^{\cdot} = \delta \varphi_l = \sum_{i=1}^{n} \frac{\partial \varphi_l}{\partial q_i} \delta q_i + \sum_{s=1}^{k} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} \delta q_s^{\cdot}$$

from which, taking into account formulas (1.6) and (1.13), we have [7]

$$\frac{d}{dt} \delta q_{k+l} - \bar{\delta} q_{k+l}^{\cdot} = \sum_{s=1}^{k} A_s^{k+l} \delta q_s \quad (l = 1, \ldots, r) \tag{1.15}$$

where the symbol $\bar{\delta}$ denotes the variation in the Appel — Suslov meaning of the function which contains dependent velocities.

Note that when on the strength of equations of motion of a nonholonomic system all $A_s^{k+l} = 0$ $(s = 1, \ldots, k)$, then the equality $d\delta q_{k+l}/dt = \delta q_{k+l}^{\cdot}$, is implied by (1.15) and vice versa. Hence when the validity of equalities (1.11) or, what is the same, equalities $A_s^{k+l} = 0$ $(s = 1, \ldots, k,\ l = 1, \ldots, r)$ is implied by the equations of motion, then in both variational methods formulas (1. 9) hold for all coordinates.

Note that in the case of linear relationships (1. 3) when

$$\varphi_l(q, q^\cdot, t) = \sum_{s=1}^{k} a_{ls}(q, t)\, q_s^\cdot + a_l(q, t) \tag{1.16}$$

the coefficients in (1. 13) are of the form [2]

$$A_s^{k+l} = \frac{da_{ls}}{dt} - \sum_{i=1}^{k} \frac{\partial a_{li}}{\partial q_s} q_i^\cdot - \frac{\partial a_l}{\partial q_s} - \sum_{j=1}^{r} a_{js} \left(\sum_{i=1}^{k} \frac{\partial a_{li}}{\partial q_{k+j}} q_i^\cdot + \frac{\partial a_l}{\partial q_{k+j}} \right)$$

and the right-hand of equality (1. 15) can be represented in the form [3]

$$\sum_{s=1}^{k} A_s^{k+l} \delta q_s = \sum_{s=1}^{k} (a_{ls}^\cdot \delta q_s - q_s^\cdot \delta a_{ls}) - \delta a_l = \delta B_l$$

Example 1.1. In Appel's example the nonlinear relationship is represented by the equation

$$f(q_1^\cdot, q_2^\cdot, q_3^\cdot) = q_3^\cdot \mp a \sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}} = 0$$

Assuming that conditions (1. 9) hold for all $i = 1, 2, 3$, in comformity with (1. 10) we have

$$\delta f = \pm a \sum_{s=1}^{2} \frac{d}{dt} \left(\frac{q_s^\cdot}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}} \right) \delta q_s$$

If, on the other hand, conditions (1. 14) are assumed valid for all $s = 1, 2$ and $\bar{\delta} f = 0$, then in conformity with (1. 15) we have

$$\frac{d}{dt} \delta q_3 - \bar{\delta} q_3^\cdot = \pm a \sum_{s=1}^{2} \frac{d}{dt} \left(\frac{q_s^\cdot}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}} \right) \delta q_s$$

i. e.

$$A_s^3 = \pm a \frac{d}{dt} \frac{q_s^\cdot}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}} \qquad (s = 1, 2)$$

Example 1.2. For a disk of radius a rolling over a rough horizontal plane the equations of nonholonomic relationships are of the form

$$f_1(\varphi, x^\cdot, \psi^\cdot) = x^\cdot + a \cos \varphi \psi^\cdot = 0, \quad f_2(\varphi, y^\cdot, \psi^\cdot) = y^\cdot + a \sin \varphi \psi^\cdot = 0$$

When conditions (1. 9) hold for all coordinates $x, y, \varphi, \theta, \psi$, then

$$\delta f_1 = a \sin \varphi (\varphi^\cdot \delta \psi - \psi^\cdot \delta \varphi), \quad \delta f_2 = a \cos \varphi (\psi^\cdot \delta \varphi - \varphi^\cdot \delta \psi)$$

If conditions (1. 14) are satisfied for $q_1 = \theta$, $q_2 = \psi$, and $q_3 = \varphi$ and conditions (1. 11) hold, then in comformity with (1. 13) we have

$$\frac{d}{dt} \delta x - \bar{\delta} x^\cdot = a \sin \varphi (\varphi^\cdot \delta \psi - \psi^\cdot \delta \varphi), \quad \frac{d}{dt} \delta y - \bar{\delta} y^\cdot = a \cos \varphi (\varphi^\cdot \delta \varphi - \varphi^\cdot \delta \psi)$$

i.e. for $q_4 = x$ and $q_5 = y$ the coefficients in (1. 13) are

$$A_1^4 = A_1^5 = 0, \quad A_2^4 = a \sin \varphi\varphi^\cdot, \quad A_3^4 = -a \sin \varphi\psi^\cdot,$$
$$A^5{}_2 = -a \cos \varphi\varphi^\cdot, \quad A_3^5 = a \cos \varphi\psi^\cdot$$

2. Let us now consider the form of Hamilton 's principle for nonholonomic systems obtained in comformity with one or the other of the described above points of view on the relation between variations of velocities and of coordinate derivatives.

Let the commutation relationships (1. 9) be satisfied for all coordinates. Substituting (1. 9) into formula (1. 8) we obtain the Hölder form [1] of Hamilton 's principle [7]

$$\int_{t_0}^{t_1} \delta L\, dt = 0 \qquad (2.1)$$

In that variational principle we compare positions of the system on the actual trajectory $q_i(t)$ with the simultaneous position obtained by moving from real motion positions by the possible displacement δq_i which define the momentarily fixed configuration of the system. The sequence of displaced positions $q_i(t) + \delta q_i$ may be considered an alternate or roundabout path which generally does not satisfy Eqs. (1. 1).

Indeed, if the alternate path satisfies Eqs. (1. 1), the equalities

$$f_l(q + \delta q, q^\cdot + \delta q^\cdot, t) = f_l(q, q^\cdot, t) + \sum_{i=1}^{n} \left(\frac{\partial f_l}{\partial q_i} \delta q_i + \frac{\partial f_l}{\partial q_i^\cdot} \delta q_i^\cdot \right) + \ldots = 0$$

hold. From these equalities follow equalities (1. 11) that are accurate to smalls of the first order. But these conditions are not satisfied, hence Hamilton 's principle (2. 1) does not generally represent the principle of stationary action [8]

$$\delta \int_{t_0}^{t_1} L\, dt = 0 \qquad (2.2)$$

as is the case of holonomic systems.

The equations of motion of nonholonomic systems are derived from Hamilton 's principle (2. 1), for example, in the form of Lagrange equations with coefficients μ_l

$$\frac{d}{dt} \frac{\partial L}{\partial q_i^\cdot} - \frac{\partial L}{\partial q_i} = \sum_{l=1}^{r} \mu_l \frac{\partial f_l}{\partial q_i^\cdot} \quad (i = 1, \ldots, n) \qquad (2.3)$$

which with Eqs. (1. 1) form a closed system of $n + r$ equations with the same number of unknowns. Note that using Eqs. (1. 1) it is possible to define the Lagrange multipliers as functions $\mu_l(q_i, q_i^\cdot, t)$, whose substitution into Eqs. (2. 3) yields a system of equations each of which is of the second order with respect to q_i. The general solution of that equation depends on the $2n$ arbitrary constants. Since Eqs. (1. 1) must also be satisfied, the number of arbitrary constants in the general solution of Eqs. (2. 3) is $2n - r$. These constants can be expressed in terms of initial data

$$q_i = q_{i0}, \quad q_i^\cdot = q_{i0}^\cdot, \quad t = t_0 \qquad (2.4)$$

by arbitrarily specifying n numbers q_{i0} that determine the position of initial points and the $n-r$ numbers $\dot{q}_{i0}$ which with Eqs. (1.1) determine the initial velocity. From this, incidentally, follows the actual property of actual trajectory of a nonholonomic system that it cannot pass through two arbitrarily specified points of space. If the initial point q_{i0} $(i = 1, \ldots, n)$ is fixed, the second point position cannot be arbitrarily specified, and it must be on a manifold of $n - r$ dimensions, which can be dynamically reached from the specified configuration, while the set of configurations, kinematically attainable from the specified configuration, is in this case of linear relationships of dimension n [8].

Let us clarify the relation between principle (2.1) and the variational principle

$$\int_{t_0}^{t_1}\left[\delta(\Theta + U) + \sum_{l=1}^{r}\frac{\partial T}{\partial \dot{q}_{k+l}}(\delta \dot{q}_{k+l} - \delta\varphi_l)\right]dt = 0 \tag{2.5}$$

used by Voronets in [2] with conditions (1.9) as one of the methods of deriving his equations of motion for nonholonomic systems of the form

$$\frac{d}{dt}\frac{\partial\Theta}{\partial \dot{q}_s} - \frac{\partial(\Theta+U)}{\partial q_s} = \sum_{l=1}^{r}\frac{\partial(\Theta+U)}{\partial q_{k+l}}\frac{\partial\varphi_l}{\partial \dot{q}_s} + \sum_{l=1}^{r}\frac{\partial T}{\partial \dot{q}_{k+l}}A_s^{k+l} \tag{2.6}$$

established by him in the case of linear relationships. In that equation $\Theta(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_k, t)$ is the kinetic energy of system $T(q, \dot{q}, t)$ from which the dependent velocities have been eliminated using formulas (1.3). Since the equalities

$$\frac{\partial\Theta}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s} + \sum_{l=1}^{r}\frac{\partial T}{\partial \dot{q}_{k+l}}\frac{\partial\varphi_l}{\partial \dot{q}_s} \quad (s = 1, \ldots, k) \tag{2.7}$$

$$\frac{\partial\Theta}{\partial q_i} = \frac{\partial T}{\partial q_i} + \sum_{l=1}^{r}\frac{\partial T}{\partial \dot{q}_{k+l}}\frac{\partial\varphi_l}{\partial q_i} \quad (i = 1, \ldots n)$$

are valid, the relation [6]

$$\delta T = \sum_{i=1}^{n}\left(\frac{\partial T}{\partial q_i}\delta q_i + \frac{\partial T}{\partial \dot{q}_i}\delta \dot{q}_i\right) = \delta\Theta + \sum_{l=1}^{r}\frac{\partial T}{\partial \dot{q}_{k+l}}(\delta \dot{q}_{k+l} - \delta\varphi_l) \tag{2.8}$$

obtained without the use of conditions (1.6), is also valid. Substituting the right-hand side of equality (2.8) for the quantity δT, appearing in $\delta L = \delta T + \delta U$, into expression (2.1), we obtain formula (2.5) which in essence is Voronets' form of Hamilton's principle for nonholonomic systems. Principle (2.5) was neither substantiated nor named in [2].

Note that each of Voronets' equations (2.6) is a second order differential equation and that they are complemented by Eqs. (1.3) of the first order. Hence the general solution of the system of Eqs. (2.6) and (1.3), as well as the general solution of system (2.3), (1.1) depend on $2k + r = 2n - r$ arbitrary constants.

We now substitute expressions (1.14) and (1.15) into formula (1.8) and, after integration by parts with allowance for conditions (1.7), obtain Hamilton's principle in Soslov's form [7]

$$\int_{t_0}^{t_1}\left(\bar{\delta} L + \sum_{l=1}^{r} \frac{\partial T}{\partial \dot{q}_{k+l}} \sum_{s=1}^{k} A_s^{k+l} \delta q_s\right) dt = 0 \qquad (2.9)$$

This formula was derived by Suslov for the case of linear relationships of the form (1.3),(1.16) and called the modification of the d'Alambert principle. He had stressed that "it in no way represents Hamilton's principle" [3], apparently meaning the principle of stationary action.

When comparing formulas (2.9) and (2.1) it should be borne in mind that the variations of Lagrange functions in these are calculated differently: in (2.1) allowance is made for equalities (1.9) and in (2.9) for equalities (1.14) and (1.15). Note also that since in the considered method of variation, conditions (1.11) are satisfied, the alternate paths $q_i(t) + \delta q_i$ in the case of (2.9) satisfy in the first approximation the conditions (1.3).

Finally, we point out that, if formula (2.8) in which, in conformity with Suslov's method of variation, it is now necessary to take into account equalities (1.11) and (1.3), which reduces it to the form $\bar{\delta} T = \delta\Theta$, then formula (2.9) assumes the form

$$\int_{t_0}^{t_1}\left[\delta(\Theta + U) + \sum_{l=1}^{r} \frac{\partial T}{\partial \dot{q}_{k+l}} \sum_{s=1}^{k} A_s^{k+l} \delta q_s\right] dt = 0 \qquad (2.10)$$

which with equalities (1.12) taken into account evidently represents Hamilton's principle in Suslov's form (2.5). We recall that Suslov had stated that "that formula (2.9) (in a somewhat different form)... was indicated by P. V. Voronets " [3].

It has been thus shown that formulas (2.1), (2.5), (2.9), and (2.10) that define Hamilton's principle for nonholonomic systems constrained by (1.3) are equivalent and convert to one another by means of transformations (2.8) and (1.12) with allowance for the relation equations and the method of variation.

E x a m p l e 2.1. In Appel's example

$$L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3$$

$$\Theta + U = \frac{m(1 + a^2)}{2}(\dot{q}_1^2 + \dot{q}_2^2) - mgq_3$$

$$\frac{\partial T}{\partial \dot{q}_3}(\delta \dot{q}_3 - \delta\varphi) = ma\sqrt{\dot{q}_1^2 + q_2^2}\,(\delta q_3^{\cdot} - a\delta\sqrt{\dot{q}_1^2 + \dot{q}_2^2})$$

$$\frac{\partial T}{\partial \dot{q}_3} \sum_{s=1}^{2} A_s^3 \delta q_s = ma^2 \sqrt{\dot{q}_1^2 + \dot{q}_2^2} \sum_{s=1}^{2} \frac{d}{dt}\left(\frac{\dot{q}_s}{\sqrt{\dot{q}_1^2 + \dot{q}_2^2}}\right)\delta q_s$$

It is seen that the relationships corresponding to formulas (2.1), (2.5), (2.9), and (2.10) defining Hamilton's principle convert to one another when the relation equation and the evident equalities

$$\dot{q}_1\delta\dot{q}_1 + \dot{q}_2\delta\dot{q}_2 = \sqrt{\dot{q}_1^2 + \dot{q}_2^2}\ \delta\sqrt{\dot{q}_1^2 + \dot{q}_2^2}, \quad \sqrt{\dot{q}_1^2 + \dot{q}_2^2}\,\bar{\delta}\dot{q}_3 = a(\dot{q}_1\delta\dot{q}_1 + \dot{q}_2\delta\dot{q}_2)$$

are taken into account.

3. Let us compare Hamilton's principle (2.1) with the Lagrange problem of stationary value of the action integral (2.2) in the class of curves that satisfy Eqs. (1.1). The introduction of indeterminate multipliers $\varkappa_l\,(t)$ reduces that problem of conditional extremum ot the problem of variations [9]

$$\delta \int_{t_0}^{t_1} \Big(L + \sum_l \varkappa_l f_l\Big)\, dt = 0 \tag{3.1}$$

The equation of Euler for problem (3.1)

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \sum_l \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right) - \sum_l \varkappa_l^{\cdot}\,\frac{\partial f_l}{\partial q_i^{\cdot}} \qquad (i = 1, \ldots, n) \tag{3.2}$$

is a second order differential equation in q_i and of first order in $\varkappa_l$. If Eqs. (1.1) are nonintegrable, the general solution of Eqs. (3.2) and (1.1) depends on $2n$ arbitrary constants, hence the equations of motion (2.3) and (1.1) in the case of a nonholonomic system are not equivalent to Eqs. (3.2) and (1.1) of the variational problem (3.1) [10, 11]. The nonequivalence of these two systems of equations does not, however, exclude the possibility of some of their solutions being the same.

Let the general or some particular solution $q_i\,(t)$ of Eqs. (2.3) and (1.1) be also the solution of Eqs. (3.2) and (1.1) for the same initial conditions (2.4). The equalities

$$\sum_l (\mu_l + \varkappa_l^{\cdot})\,\frac{\partial f_l}{\partial q_i^{\cdot}} = \sum_l \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right) \qquad (i = 1, \ldots, n) \tag{3.3}$$

are now evidently valid.

Taking into account (1.5) we multiply Eqs. (3.3) by the possible permutations of δq_i and summating over all i's, we obtain the condition

$$\sum_{l,\,i} \varkappa_l \left(\frac{\partial f_l}{\partial q_i} - \frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right) \delta q_i = 0 \tag{3.4}$$

which is necessary if the two systems are to have the same solution $q_i(t)$. This condition is also sufficient. To prove this, let us assume that some solution of Eqs. (3.2) and (1.1) satisfies condition (3.4) for any δq_i compatible with conditions (1.5). Multiplying Eqs. (3.2) by possible permutations of δq_i and Eqs. (1.5) by the indeterminate multipliers μ_l, and summating over all i's and l's with allowance for (3.4) and (1.5) we obtain the relationship

$$\sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} - \sum_l \mu_l\,\frac{\partial f_l}{\partial q_i^{\cdot}}\right) \delta q_i = 0$$

which shows that the considered solution $q_i(t)$ also satisfies Eqs. (2.3) and (1.1).

Thus condition (3.4) is necessary and sufficient for solution $q_i\,(t)$ of Eqs. (2.3) and (1.1) to be among solutions of Eqs. (3.2) and (1.1) [5].

Note that equalities (3.3) follow from condition (3.4). This can be proved by multiplying conditions (1.5) by multipliers $\mu_l + \varkappa_l^{\cdot}$ where μ_l are the indeterminate multipliers, then summating over all l's, subtracting from (3.4) and reducing to zero the coefficients at dependent variations by suitable selection of multipliers μ_l. As the result we obtain the equalities (3.3).

Thus, when conditions (3.4) are satisfied, the equations of motion (2.3) of a nonholonomic system have the form of Euler's equations (3.2). Owing to this we say that Hamilton's principle (2.1) for the motion of a nonholonomic system defined by such solutions has the characteristics of the principle of stationary action (2.2).

Note also that when in problem (3.1) the symbol δ is understood to represent variations in the class of possible permutations (1.5) of a nonholonomic system, then (2.2) coincides with (3.1), provided however that condition

$$\int_{t_0}^{t_1} \sum_l \varkappa_l \delta f_l \, dt = 0$$

is satisfied, which happens then and only then when condition (3.4) is satisfied with (1.10) taken into account.

For relationships of the form (1.3) equality (3.4) reduces to conditions

$$\sum_l \varkappa_l A_s^{k+l} = 0 \quad (s = 1, \ldots, k) \tag{3.5}$$

A particular form of conditions (3.5) in the problem of rolling a heavy solid body on a horizontal absolutely rough plane appeared in [11].

It has thus been shown that for Hamilton's principle (2.1) for a nonholonomic system to have the characteristics of the principle of stationary action it is necessary and sufficient that condition (3.4) or, what is the same, condition (3.5) is satisfied.

Note that Hamilton's principle in the form (2.9) obtained by Suslov has also the characteristics of the principle of stationary action (2.2) then and only then when the condition

$$\int_{t_0}^{t_1} \sum_{l,s} \frac{\partial T}{\partial \dot{q}_{k+l}} A_s^{k+l} \delta q_s \, dt = 0$$

is satisfied. Since in that formula δq_s are arbitrary and independent, it is satisfied only when

$$\sum_l \frac{\partial T}{\partial \dot{q}_{k+l}} A_s^{k+l} = 0 \quad (s = 1, \ldots, k) \tag{3.6}$$

For motions that satisfy conditions (3.6) Voronets' equations (2.6) are of the form of equations of motion of nonholonomic systems.

We stress that conditions (3.4) – (3.6) are seldom satisfied in the case of nonholonomic systems. Two examples are given below. In the first, these conditions are satisfied by the general solution, and in the second only for some particular solutions of equations of motion of a nonholonomic system (in the latter Hamilton's principle has the characteristics of the principle of stationary action, not for all but only for related motions of the nonholonomic system). Examples can be given of nonholonomic system for which these conditions are generally not satisfied [8].

Example 3.1. In Appel 's example (Example 1.1) from equations of the form of (2.3) and (1.1)

$$mq_1^{\cdot\cdot} = -\mu a \frac{q_1^{\cdot}}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}}, \qquad mq_2^{\cdot\cdot} = -\mu a \frac{q_2^{\cdot}}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}},$$

$$mq_3^{\cdot\cdot} = -mg + \mu$$

we have

$$\frac{d}{dt}\left(\frac{q_s^{\cdot}}{\sqrt{q_1^{\cdot 2} + q_2^{\cdot 2}}}\right) = 0 \quad (s = 1, 2)$$

which shows that conditions (3.4) – (3.6) are satisfied for all motions of the point.

Example 3.2. For a disk (Example 1.2) the Lagrange function is

$$L = \frac{m}{2}\{[x^{\cdot} - a(\cos\theta\sin\varphi\theta^{\cdot} + \sin\theta\cos\varphi\varphi^{\cdot})]^2 +$$

$$[y^{\cdot} + a(\cos\theta\cos\varphi\theta^{\cdot} - \sin\theta\sin\varphi\varphi^{\cdot})]^2\} +$$

$$\frac{1}{2}A(\theta^{\cdot 2} + \varphi^{\cdot 2}\cos^2\theta) + \frac{1}{2}C(\psi^{\cdot} + \varphi^{\cdot}\sin\theta)^2 - mga\cos\theta$$

Taking into account relationships, conditions (3.6) assume the form

$$\sum_{l=1}^{2} \frac{\partial T}{\partial q_{3+l}^{\cdot}} A_2^{3+l} = -ma^2\theta^{\cdot}\varphi^{\cdot}\cos\theta = 0, \qquad \sum_{l=1}^{2} \frac{\partial T}{\partial q_{3+l}^{\cdot}} A_3^{3+l} = ma^2\theta^{\cdot}\psi^{\cdot}\cos\theta = 0$$

which are satisfied either when $\theta^{\cdot} = 0$ or $\varphi^{\cdot} = \psi^{\cdot} = 0$. It can be shown [11] that then conditions (3.5) are also satisfied. Hence in the case of motion of a disk whose plane is at constant angle to the vertical, as well as some highly special motions for which $\varphi^{\cdot} = \psi^{\cdot} = 0$, Hamilton 's principle has the characteristics of the principle of stationary motion, while generally this is not so.

4. The use of Hamilton 's principle for determining the stationarity of action in the case of real motions is closely related to the problem of extending to nonholonomic systems the Hamilton – Jacobi method of integration of canonical equations of motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + \sum_l \mu_l \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{4.1}$$

that are equivalent to Eqs. (2.3) and (1.1).

As usual, the generalized momenta

$$p_i = \partial L / \partial q_i^{\cdot} \quad (i = 1, \ldots, n)$$

and the Hamiltonian

$$H(q, p, t) = \sum_{i=1}^{n} p_i q_i^{\cdot} - L \tag{4.2}$$

In essence the Hamilton – Jacobi method consists of the following [4.5]. Variables

$$\pi_i = p_i + \sum_{l=1}^{r} \lambda_l \frac{\partial f_l}{\partial q_i^{\cdot}} \quad (i = 1, \ldots, n) \tag{4.3}$$

are introduced and used for reducing formula (4.2) to the form

$$L = \sum_{i=1}^{n} \pi_i q_i^{\cdot} - H_1 \tag{4.4}$$

where function

$$H_1(q, \pi, t) = H(q, p, t) + \sum_{l,i} \lambda_l \frac{\partial f_l}{\partial q_i^{\cdot}} q_i^{\cdot} \tag{4.5}$$

is obtained by the substitution into its right-hand side of functions $p_i(q, \pi, t)$, and $\lambda_l(q, \pi, t)$ derived from Eqs. (1.1) and (4.3) and of the first group of Eqs. (4.1).

It is advisable to construct function (4.5) as follows. Using (1.3) represent the Lagrange function in the form $L^*(q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_k^{\cdot}, t) = \Theta + U$ and introduce in the analysis the generalized momenta and the Hamiltonian

$$P_s = \frac{\partial L^*}{\partial q_s^{\cdot}} = p_s + \sum_{l=1}^{r} p_{k+l} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} \quad (s = 1, \ldots, k) \tag{4.6}$$

$$H^*(q, P, t) = \sum_{s=1}^{k} P_s q_s^{\cdot} - L^* \tag{4.7}$$

Function (4.7) is related to function (4.2) by the formula

$$H^*(q, P, t) = H(q, p, t) + \sum_{l=1}^{r} p_{k+l} \left(\sum_{s=1}^{k} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} q_s^{\cdot} - \varphi_l \right) \tag{4.8}$$

Since equalities (4.3) imply for relationships (1.3) the equalities

$$\lambda_l = \pi_{k+l} - p_{k+l}, \quad P_s = \pi_s + \sum_{l} \pi_{k+l} \frac{\partial \varphi_l}{\partial q_s^{\cdot}}$$

function (4.5) with allowance for (4.8) assumes the form

$$H_1(q, \pi, t) = H^*(q, P, t) + \sum_{l} \pi_{k+l} \left(\varphi_l - \sum_{s} \frac{\partial \varphi_l}{\partial q_s^{\cdot}} q_s^{\cdot} \right)$$

The generalized Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H_1\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0 \tag{4.9}$$

is an equation in partial derivatives of the first order whose characteristic equations are of the canonical form

$$\frac{dq_i}{dt}=\frac{\partial H_1}{\partial \pi_i},\quad \frac{d\pi_i}{dt}=-\frac{\partial H_1}{\partial q_i}\quad (i=1,\ldots,n) \tag{4.10}$$

According to Jacobi's theorem the relationships

$$\partial S/\partial q_i=\pi_i,\quad \partial S/\partial \alpha_i=\beta_i\quad (i=1,\ldots,n)$$

represent $2n$ integrals of Eqs.(4.10) if $S(q_i,\alpha_i,t)$ is a complete integral of Eq. (4.12) with arbitrary constants α_i and β_i.

It was shown in [5] that the solution of Eqs. (4.10) is also a solution of equations of motion (4.1) if and only if it satisfies the condition

$$\sum_{l,i}\lambda_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right)\delta q_i=0 \tag{4.11}$$

Hence (4.11) is a necessary and sufficient condition for the considered generalized Hamilton —Jacobi method to be applicable to nonholonomic systems.

Note that another form of the necessary and sufficient conditions of applicability to nonholonomic systems of the potential method of integration was proposed by Arzhanykh [13].

Condition (4.11) with (1.5) taken into account follows from equations [5]

$$\frac{dp_i}{dt}+\frac{\partial H}{\partial q_i}=\sum_l\lambda_l\left(\frac{\partial f_l}{\partial q_i}-\frac{d}{dt}\frac{\partial f_l}{\partial q_i^{\cdot}}\right)-\sum_l\lambda_l^{\cdot}\frac{\partial f_l}{\partial q_i^{\cdot}}\quad (i=1,\ldots,n) \tag{4.12}$$

obtained by differentiating expressions (4.3) with respect to t on the basis of (4.10). When $\lambda_l=\varkappa_l\ (l=1,\ldots,r)$ Eqs. (4.12) evidently match Euler's equations (3.2) of the variational problem (3.1).

Hence the generalized Hamilton —Jacobi method of integrating Eqs. (4.1) of motion of nonholonomic systems is applicable if and only if Hamilton's principle has the characteristics of the principle of stationary action.

Motions of a nonholonomic system that satisfy condtions (4.11) are defined by Hamilton's canonical equations (4.10) whose corollary is the principle of stationary action

$$\delta\int_{t_0}^{t_1}\left(\sum_i\pi_i q_i^{\cdot}-H_1\right)dt=0$$

which with allowance for (4.4) is equivalent to the principle (2.2).

Example 4.1. The equations of motion in Appel's example [4] and the equations of motion of disk in the case of $\theta^{\cdot}=0$ were integrated in [14] by the Hamilton —Jacobi method. It can be shown that condition (4.11) obtained here is not satisfied for a gyroscope suspended in gimbals and constrained by the relationship (3.86) described in [4]. Because of this the solution proposed in [4] does not satisfy the equations of motion of a gyroscope.

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